

# The Number of Convex Polyominoes and the Generating Function of Jacobi Polynomials

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**Abstract.** Lin and Chang gave a generating function of convex polyominoes with an  $m + 1$  by  $n + 1$  minimal bounding rectangle. Gessel showed that their result implies that the number of such polyominoes is

$$\frac{m+n+mn}{m+n} \binom{2m+2n}{2m} - \frac{2mn}{m+n} \binom{m+n}{m}^2.$$

We show that this result can be derived from some binomial coefficients identities related to the generating function of Jacobi polynomials.

Some (binomial coefficients) identities arise from alternative solutions of combinatorial problems and incidentally give added significance to doing problems the “hard” way.

— J. RIORDAN

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## 1 Introduction

A *polyomino* is a connected union of squares in the plane whose vertices are lattice points. A polyomino is called *convex* if its intersection with any horizontal or vertical line is either empty or a line segment. Any convex polyomino has a minimal bounding rectangle whose perimeter is the same as that of the polyomino. Delest and Viennot [4] found a generating function for counting convex polyominoes by perimeter and showed that the number of convex polyominoes with perimeter  $2n + 8$ , for  $n \geq 0$ , is

$$(2n + 11)4^n - 4(2n + 1) \binom{2n}{n}. \quad (1)$$

Later, Lin and Chang [9] gave a generating function for the number of convex polyominoes with an  $(m+1) \times (n+1)$  minimal bounding rectangle, and Gessel [6] showed that their result implies that the number of such polyominoes is

$$\frac{m+n+mn}{m+n} \binom{2m+2n}{2m} - \frac{2mn}{m+n} \binom{m+n}{m}^2, \quad (2)$$

which is easily seen to give a refinement of Delest and Viennot's formula.

Since Gessel [6] (see also Bousquet-Mélou [2]) derived (2) from the generating function of Lin and Chang [9] (see also Bousquet-Mélou and Guttman [3]), it would be interesting to find an independent proof of (2). For Delest and Viennot's formula (1) such a proof was already given by Kim [8]. The aim of this paper is to provide such a proof for (2) by generalizing Kim's elementary approach. It turns out that the resulting binomial coefficients identities are related to the generating function of Jacobi polynomials.

In the next section, we translate the enumeration of convex polyominoes with fixed minimal bounding rectangle as that of two pairs of non intersecting lattice paths, which results to evaluate a quadruple sum of binomial coefficients. In Section 3, we establish some binomial coefficients identities which lead to the evaluation of the desired sums.

## 2 Non intersecting lattice paths and determinant formula

A *lattice path* is a sequence of points  $(s_0, s_1, \dots, s_n)$  in the plan  $\mathbb{Z}^2$  such that either  $s_i - s_{i-1} = (1, 0)$ ,  $(0, 1)$  for all  $i = 1, \dots, n$  or  $s_i - s_{i-1} = (1, 0)$ ,  $(0, -1)$  for all  $i = 1, \dots, n$ . Let  $\mathcal{P}_{m,n}$  be the set of convex polyominoes with an  $m+1$  by  $n+1$  minimal bounding rectangle. As illustrated in Figure 1, any polyomino in  $\mathcal{P}_{m,n}$  can be characterized by 4 lattice paths  $L_1, L_2, L_3$  and  $L_4$  which are given by

$$\begin{aligned} L_1: & (0, b_1) \longrightarrow (a_1, 0), \\ L_2: & (m+1-a_2, n+1) \longrightarrow (m+1, n+1-b_2), \\ L_3: & (a_1+1, 0) \longrightarrow (m+1, n-b_2), \\ L_4: & (0, b_1+1) \longrightarrow (m-a_2, n+1). \end{aligned}$$

Note that a polyomino in  $\mathcal{P}_{m,n}$  is convex if and only if the two lattice paths  $L_1, L_2$  (resp.  $L_3, L_4$ ) don't intersect. The following lemma can be readily proved by switching the *tails* of two lattice paths, which is also a special case of a more general result [7].

**Lemma 1** *Let  $a, b, c$  and  $d$  be non negative integers such that  $a' > a$ ,  $b > b'$ ,  $c > a$ ,  $d > b$ ,  $c' > a'$  and  $d' > b'$ . Then the number of pairs of non intersecting lattice paths  $(\mathcal{P}_1, \mathcal{P}_2)$  such that  $\mathcal{P}_1: (a, b) \longrightarrow (c, d)$  and  $\mathcal{P}_2: (a', b') \longrightarrow (c', d')$  is given by*

$$\binom{c-a+d-b}{c-a} \binom{c'-a'+d'-b'}{c'-a'} - \binom{c-a'+d-b'}{c-a'} \binom{c'-a+d'-b}{c'-a}.$$

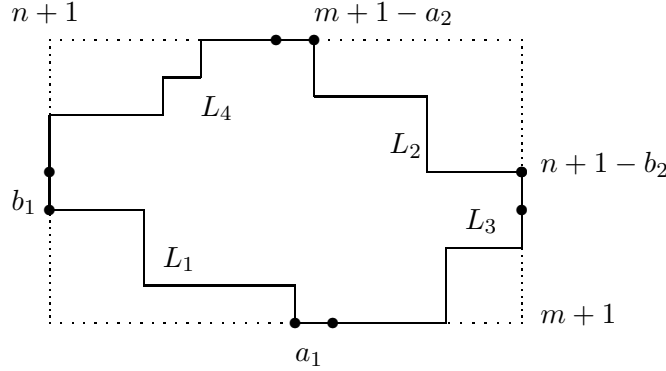


Figure 1: A convex polyomino with an  $m + 1$  by  $n + 1$  minimal bounding rectangle.

It follows that the cardinality of  $\mathcal{P}_{m,n}$  is given by

$$\sum_{a_1, a_2=0}^m \sum_{b_1, b_2=0}^n \left[ \binom{a_1 + b_1 - 2}{a_1 - 1} \binom{a_2 + b_2 - 2}{a_2 - 1} - \binom{a_1 + a_2 + n - m - 2}{n - 1} \binom{b_1 + b_2 + m - n - 2}{m - 1} \right] \cdot \left[ \binom{m + n - a_2 - b_1}{m - a_2} \binom{m + n - a_1 - b_2}{m - a_1} - \binom{m + n - a_1 - a_2}{n + 1} \binom{m + n - b_1 - b_2}{m + 1} \right], \quad (3)$$

Note that in (3) we have adopted the convention that  $\binom{-2}{-1} = 1$ , which corresponds to  $a_1 = b_1 = 0$  or  $a_2 = b_2 = 0$ . In this case the path  $L_1$  or  $L_2$  is a point.

We next split the sum in (3) into three terms: the  $a_1 = a_2 = b_1 = b_2 = 0$  term,

$$S_0 = \binom{m + n}{m}^2 - \binom{m + n}{m - 1} \binom{m + n}{n - 1}, \quad (4)$$

the  $a_1 = b_1 = 0$  or  $a_2 = b_2 = 0$  terms,

$$S_1 = 2 \sum_{a=1}^m \sum_{b=1}^n \binom{a + b - 2}{a - 1} \left[ \binom{m + n - a}{m - a} \binom{m + n - b}{m} - \binom{m + n - a}{n + 1} \binom{m + n - b}{m + 1} \right],$$

and the sum in (3) for  $a_i$  and  $b_i \geq 1$ . This third term can be split into four more terms obtained from the product:  $(a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) = S_2 - S_3 - S_4 + S_5$ . The last sum  $S_5$  is 0 because the sum of the numerator parameters of binomial coefficients are less than that of the denominator parameters.

We now proceed to evaluate or simplify  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$  using the Chu-Vandermonde formula:

$${}_2F_1 \left( \begin{matrix} -n, & a \\ & c \end{matrix} ; 1 \right) := \sum_{k \geq 0} \frac{(-n)_k (a)_k}{(c)_k k!} = \frac{(c - a)_n}{(c)_n},$$

where  $(x)_n = x(x + 1) \dots (x + n - 1)$  for  $n \geq 1$  and  $(x)_0 = 1$ .

- Applying the Chu-Vandermonde formula to the  $b$ -sums for  $S_1$  yields

$$S_1 = 2 \sum_{a=1}^m \left[ \binom{m + n - a}{n} \binom{m + n + a - 1}{n - 1} - \binom{m + n - a}{n + 1} \binom{m + n + a - 1}{n - 2} \right].$$

As

$$\begin{aligned} & \binom{m+n-a}{n} \binom{m+n+a-1}{n-1} - \binom{m+n-a}{n+1} \binom{m+n+a-1}{n-2} \\ &= \binom{m+n-a+1}{n+1} \binom{m+n+a-1}{n-1} - \binom{m+n-a}{n+1} \binom{m+n+a}{n-1}, \end{aligned}$$

by telescoping it follows that

$$S_1 = 2 \binom{m+n}{n+1} \binom{m+n}{n-1}. \quad (5)$$

- Consider now the second sum  $S_2$ :

$$S_2 = \sum_{a_1, a_2=1}^m \sum_{b_1, b_2=1}^n \binom{a_1+b_1-2}{a_1-1} \binom{a_2+b_2-2}{a_2-1} \binom{m+n-a_2-b_1}{m-a_2} \binom{m+n-a_1-b_2}{m-a_1}.$$

By the Chu-Vandermonde formula we have

$$\begin{aligned} \sum_{b_1=1}^n \binom{a_1+b_1-2}{a_1-1} \binom{m+n-a_2-b_1}{m-a_2} &= \binom{m+n+a_1-a_2-1}{n-1}, \\ \sum_{b_2=1}^n \binom{a_2+b_2-2}{a_2-1} \binom{m+n-a_1-b_2}{m-a_1} &= \binom{m+n-a_1+a_2-1}{n-1}. \end{aligned}$$

Hence

$$S_2 = \sum_{a_1, a_2=1}^m \binom{m+n+a_1-a_2-1}{n-1} \binom{m+n-a_1+a_2-1}{n-1}.$$

Setting  $a = a_1 - a_2$  we can rewrite the above sum as

$$\begin{aligned} S_2 &= \sum_{a=1-m}^{m-1} \#\{(a_1, a_2) \in [1, m]^2 \mid a_1 - a_2 = a\} \binom{m+n+a-1}{n-1} \binom{m+n-a-1}{n-1} \\ &= \sum_{a=-m}^m (m - |a|) \binom{m+n+a-1}{n-1} \binom{m+n-a-1}{n-1} \\ &= m \sum_{a=-m}^m \binom{m+n+a-1}{n-1} \binom{m+n-a-1}{n-1} \\ &\quad - 2 \sum_{a=1}^m a \binom{m+n+a-1}{n-1} \binom{m+n-a-1}{n-1}. \end{aligned}$$

By the Chu-Vandermonde formula we have

$$m \sum_{a=-m}^m \binom{m+n+a-1}{n-1} \binom{m+n-a-1}{n-1} = m \binom{2m+2n-1}{2n-1}.$$

Since

$$\begin{aligned} & 2a \binom{m+n+a-1}{n-1} \binom{m+n-a-1}{n-1} \\ &= n \binom{m+n+a-1}{n} \binom{m+n-a}{n} - n \binom{m+n+a}{n} \binom{m+n-a-1}{n}, \end{aligned}$$

telescoping yields

$$\sum_{a=1}^m 2a \binom{m+n+a-1}{n-1} \binom{m+n-a-1}{n-1} = n \binom{m+n}{n} \binom{m+n-1}{n}.$$

Hence

$$S_2 = \frac{mn}{m+n} \binom{2m+2n}{2m} - \frac{mn}{m+n} \binom{m+n}{m}^2. \quad (6)$$

- Look at the term  $S_3$ :

$$\begin{aligned} S_3 &= \sum_{a_1, a_2=1}^m \sum_{b_1, b_2=1}^n \binom{a_1 + a_2 + n - m - 2}{n-1} \binom{b_1 + b_2 + n - m - 2}{m-1} \\ &\quad \cdot \binom{m+n-a_2-b_1}{m-a_2} \binom{m+n-a_1-b_2}{m-a_1}. \end{aligned}$$

Summing the  $a_2$ -sum and  $b_2$ -sum by the Chu-Vandermonde formula yields

$$\begin{aligned} \sum_{a_2=1}^m \binom{a_1 + a_2 + n - m - 2}{n-1} \binom{m+n-a_2-b_1}{m-a_2} &= \binom{2n+a_1-b_1-1}{a_1-1}, \\ \sum_{b_2=1}^n \binom{b_1 + b_2 + n - m - 2}{m-1} \binom{m+n-a_1-b_2}{m-a_1} &= \binom{2m-a_1+b_1-1}{b_1-1}. \end{aligned}$$

Hence, replacing  $a_1$  and  $b_1$  by  $a$  and  $b$  respectively we get

$$\begin{aligned} S_3 &= \sum_{a=1}^m \sum_{b=1}^n \binom{2m-a+b-1}{b-1} \binom{2n+a-b-1}{a-1} \\ &= \sum_{a=1}^m \sum_{b=1}^n \binom{m+n+a-b-1}{m+a-1} \binom{m+n-a+b-1}{n+b-1}, \end{aligned}$$

by the substitutions  $a \leftarrow m - a + 1$  and  $b \leftarrow n - b + 1$ .

- Finally we have

$$S_4 = \sum_{a_1, a_2=1}^m \sum_{b_1, b_2=1}^n \binom{a_1 + b_1 - 2}{a_1 - 1} \binom{a_2 + b_2 - 2}{a_2 - 1} \binom{m+n-a_1-a_2}{n+1} \binom{m+n-b_1-b_2}{m+1}.$$

Summing the  $a_1$ -sum and  $b_2$ -sum by the Chu-Vandermonde formula yields

$$\sum_{a_1=1}^m \binom{a_1 + b_1 - 2}{a_1 - 1} \binom{m + n - a_1 - a_2}{n + 1} = \binom{m + n - a_2 + b_1 - 1}{n + b_1 + 1},$$

$$\sum_{b_2=1}^n \binom{a_2 + b_2 - 2}{a_2 - 1} \binom{m + n - b_1 - b_2}{m + 1} = \binom{m + n + a_2 - b_1 - 1}{m + a_2 + 1}.$$

Substituting  $a_2$  and  $b_1$  by  $a$  and  $b$  we obtain

$$S_4 = \sum_{a=1}^{m-2} \sum_{b=1}^{n-2} \binom{m + n + a - b - 1}{m + a + 1} \binom{m + n - a + b - 1}{n + b + 1},$$

for the summand is zero if  $a = m - 1, m$  or  $b = n - 1, n$ .

We shall evaluate  $S_3$  and  $S_4$  in the next section.

### 3 Jacobi polynomials and evaluation of $S_3$ and $S_4$

Set

$$\Delta := \sqrt{1 - 2x - 2y - 2xy + x^2 + y^2}.$$

The following identity is equivalent to the generating function of Jacobi polynomials:

$$\sum_{m,n=0}^{\infty} \binom{m + n + \alpha}{m} \binom{m + n + \beta}{n} x^m y^n = \frac{2^{\alpha+\beta}}{\Delta(1 - x + y + \Delta)^{\alpha}(1 + x - y + \Delta)^{\beta}}. \quad (7)$$

The reader is referred to [1, p. 298] and [10, p. 271] for two classical analytical proofs and to [5] for a combinatorial proof.

Applying the operator  $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2$  to the  $\alpha = \beta = 1$  case of (7) yields:

$$\sum_{m,n \geq 1} \frac{m + n}{2} \binom{m + n - 1}{m} \binom{m + n - 1}{n} x^m y^n = \frac{xy}{\Delta^3}. \quad (8)$$

**Theorem 2** *We have*

$$S_3 = \frac{mn}{2(m + n)} \binom{m + n}{m}^2. \quad (9)$$

*Proof.* Consider the generating function of  $S_4$ :

$$\begin{aligned} F(x, y) &:= \sum_{m,n=0}^{\infty} \sum_{a=1}^m \sum_{b=1}^n \binom{m + n - a + b - 1}{m - a} \binom{m + n + a - b - 1}{n - b} x^m y^n \\ &= \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} x^a y^b \sum_{m=a}^{\infty} \sum_{n=b}^{\infty} \binom{m + n - a + b - 1}{m - a} \binom{m + n + a - b - 1}{n - b} x^{m-a} y^{n-b} \\ &= \sum_{a,b=1}^{\infty} x^a y^b \sum_{m,n=0}^{\infty} \binom{m + n + 2b - 1}{m} \binom{m + n + 2a - 1}{n} x^m y^n. \end{aligned}$$

Applying (7) to the inner double sum yields

$$F(x, y) = \sum_{a,b=1}^{\infty} x^a y^b \frac{2^{2a+2b-2}}{\Delta(1-x+y+\Delta)^{2b-1}(1+x-y+\Delta)^{2a-1}} = \frac{xy}{\Delta^3}.$$

The theorem follows then from (8). ■

**Theorem 3** *There holds*

$$S_4 = \binom{m+n}{m}^2 + \binom{m+n}{m-1} \binom{m+n}{n-1} + \frac{mn}{2(m+n)} \binom{m+n}{m}^2 - \binom{2m+2n}{2n}. \quad (10)$$

*Proof.* Consider the generating function of  $S_4$ :

$$\begin{aligned} G(x, y) &:= \sum_{m,n=0}^{\infty} \sum_{a=1}^{m-2} \sum_{b=1}^{n-2} \binom{m+n-a+b-1}{m-a-2} \binom{m+n+a-b-1}{n-b-2} x^m y^n \\ &= \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{m=a+2}^{\infty} \sum_{n=b+2}^{\infty} \binom{m+n-a+b-1}{m-a-2} \binom{m+n+a-b-1}{n-b-2} x^m y^n \\ &= \sum_{a,b=1}^{\infty} x^{a+2} y^{b+2} \sum_{m,n=0}^{\infty} \binom{m+n+2b+3}{m} \binom{m+n+2a+3}{n} x^m y^n. \end{aligned}$$

Applying (7) to the inner double sum yields

$$\begin{aligned} G(x, y) &= \sum_{a,b=1}^{\infty} x^{a+2} y^{b+2} \frac{2^{2a+2b+6}}{\Delta(1-x+y+\Delta)^{2b+3}(1+x-y+\Delta)^{2a+3}} \\ &= \frac{16x^3 y^3}{\Delta^3(1-x-y+\Delta)^4}. \end{aligned}$$

Set

$$f(x, y) := \sum_{m,n=0}^{\infty} \binom{m+n}{m} x^m y^n = \frac{1}{1-x-y}.$$

By bisecting twice, we get the terms of even powers of  $x$  and  $y$  in  $f(x, y)$ :

$$\sum_{m,n=0}^{\infty} \binom{2m+2n}{2m} x^{2m} y^{2n} = \frac{1}{4} (f(x, y) + f(-x, y) + f(x, -y) + f(-x, -y))$$

i.e.,

$$\sum_{m,n=0}^{\infty} \binom{2m+2n}{2m} x^m y^n = \frac{1-x-y}{\Delta^2}.$$

Now, the  $\alpha = \beta = 0$  and  $\alpha = \beta = 2$  cases of (7) read:

$$\begin{aligned} \sum_{m,n=0}^{\infty} \binom{m+n}{m}^2 x^m y^n &= \frac{1}{\Delta}, \\ \sum_{m,n=1}^{\infty} \binom{m+n}{m-1} \binom{m+n}{n-1} x^m y^n &= \frac{4xy}{\Delta(1-x-y+\Delta)^2}. \end{aligned}$$

As

$$\frac{16x^3y^3}{\Delta^3(1-x-y+\Delta)^4} = \frac{1}{\Delta} + \frac{4xy}{\Delta(1-x-y+\Delta)^2} + \frac{xy}{\Delta^3} - \frac{1-x-y}{\Delta^2},$$

extracting the coefficients of  $x^m y^n$  in the above equation completes the proof. ■

Summarizing, formula (2) follows then from (4)–(6), (9) and (10).

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